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On lacunary double statistical convergence in locally solid Riesz spaces

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Abstract

The concept of statistical convergence is one of the most active areas of research in the field of summability. Most of the new summability methods have relation with this popular method. In this paper, we introduce the concept of double $\mathcal{I}_\theta$-statistical-τ-convergence which is a more general idea of statistical convergence. We also investigate the ideas of double $\mathcal{I}_\theta$-statistical-τ-boundedness and double $\mathcal{I}_\theta$-statistical-τ-Cauchy condition of sequences in the framework of locally solid Riesz space endowed with a topology τ and investigate some of their consequences.

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Keywords: ideal; filter; double $\mathcal{I}$-statistical-τ-convergence; double $\mathcal{I}_\theta$-statistical-τ-convergence; double $\mathcal{I}_\theta$-statistical-τ-boundedness; double $\mathcal{I}_\theta$-statistical-τ-Cauchy condition

1 Introduction

The notion of statistical convergence, which is an extension of the idea of usual convergence, was introduced by Fast [1], Steinhaus [2] independently in the same year 1951 and also by Schoenberg [3]. Its topological consequences were studied first by Fridy [4] and Šalát [5]. The notion has also been defined and studied in different steps, for example, in a locally convex space [6]; in topological groups [7, 8]; in probabilistic normed spaces [9, 10], in intuitionistic fuzzy normed spaces [11], in random 2-normed spaces [12]. In [13] Albayrak and Pehlivan studied this notion in locally solid Riesz spaces. Recently, Mohiuddine *et al.* [14] studied statistically convergent, statistically bounded and statistically Cauchy for double sequences in locally solid Riesz spaces. Also, in [15] Mohiuddine *et al.* introduced the concept of lacunary statistical convergence, lacunary statistically bounded and lacunary statistically Cauchy in the framework of locally solid Riesz spaces. Quite recently, Das and Savas [16] introduced the ideas of $\mathcal{I}_\tau$-convergence, $\mathcal{I}_\tau$-boundedness and $\mathcal{I}_\tau$-Cauchy condition of nets in a locally solid Riesz space.

The more general idea of lacunary statistical convergence was introduced by Fridy and Orhan in [17]. Subsequently, a lot of interesting investigations have been done on this convergence (see, for example, [18–21] where more references can be found).

The idea of statistical convergence was further extended to $\mathcal{I}$-convergence in [22] using the notion of ideals of $\mathbb{N}$ with many interesting consequences. More investigations in this direction and more applications of ideals can be found in [22–31] where many important references can be found.

Recently in [24, 30] we used ideals to introduce the concepts of $\mathcal{I}^{\lambda}$-statistical convergence and $\mathcal{I}$-lacunary-statistical convergence and investigated their properties.

The notion of a Riesz space was first introduced by Riesz [32] in 1928, and since then it has found several applications in measure theory, operator theory, optimization and also in economics (see [33]). It is well known that a topology on a vector space that makes the operations of addition and scalar multiplication continuous is called a linear topology and a vector space endowed with a linear topology is called a topological vector space. A Riesz space is an ordered vector space which is also a lattice endowed with a linear topology. Further, if it has a base consisting of solid sets at zero, then it is known as a locally solid Riesz space.

In this paper, we introduce the idea of $\mathcal{I}$-double lacunary statistical convergence in a locally solid Riesz space and study some of its properties by using the mathematical tools of the theory of topological vector spaces.

2 Preliminaries

We now recall the following basic facts from [22].

A family $\mathcal{I}$ of subsets of a non-empty set X is said to be an ideal if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (ii) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$. $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\phi\}$ and $X \notin \mathcal{I}$. $\mathcal{I}$ is admissible if it contains all singletons. If $\mathcal{I}$ is a proper non-trivial ideal, then the family of sets $F(\mathcal{I}) = \{M \subset X : M^c \in \mathcal{I}\}$ is a filter on X (where c stands for the complement). It is called the filter associated with the ideal $\mathcal{I}$.

We also recall some of the basic concepts of Riesz spaces.

Definition 2.1 Let L be a real vector space and let $\leq$ be a partial order on this space. L is said to be an ordered vector space if it satisfies the following properties:

(i) If $x, y \in L$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in L$.

(ii) If $x, y \in L$ and $y \leq x$, then $\lambda y \leq \lambda x$ for each $\lambda \geq 0$.

If in addition L is a lattice with respect to the partial ordering, then L is said to be a Riesz space (or a vector lattice).

For an element x of a Riesz space L, the positive part of x is defined by $x^+ = x \vee \theta$, the negative part of x by $x^- = (-x) \vee \theta$ and the absolute value of x by $|x| = x \vee (-x)$, where θ is the element zero of L.

A subset S of a Riesz space L is said to be solid if $y \in S$ and $|x| \leq |y|$ imply $x \in S$.

A topology τ on a real vector space L that makes the addition and scalar multiplication continuous is said to be a linear topology, that is, when the mappings

$$(x, y) \to x + y \quad \bigl(\text{from } (L \times L, \tau \times \tau) \to (L, \tau)\bigr),$$
$$(\lambda, x) \to \lambda x \quad \bigl(\text{from } (R \times L, \sigma \times \tau) \to (L, \tau)\bigr)$$

are continuous, where σ is the usual topology on R. In this case, the pair (L, τ) is called a topological vector space.

Every linear topology τ on a vector space L has a base $\mathcal{N}$ for the neighborhoods of θ satisfying the following properties:

(a) Each $V \in \mathcal{N}$ is a balanced set, that is, $\lambda x \in V$ holds for all $x \in V$ and every $\lambda \in R$ with $|\lambda| \leq 1$.

(b) Each $V \in \mathcal{N}$ is an absorbing set, that is, for every $x \in L$, there exists a $\lambda > 0$ such that $\lambda x \in V$.

(c) For each $V \in \mathcal{N}$, there exists some $W \in \mathcal{N}$ with $W + W \subset V$.

Definition 2.2 A linear topology τ on a Riesz space L is said to be locally solid if τ has a base at zero consisting of solid sets. A locally solid Riesz space (L, τ) is a Riesz space L equipped with a locally solid topology τ.

$\mathcal{N}_{\text{sol}}$ will stand for a base at zero consisting of solid sets and satisfying the properties (a), (b) and (c) in a locally solid topology.

3 Main results

The notion of statistical convergence depends on the density of subsets of $\mathbf{N}$, the set of natural numbers. A subset E of $\mathbf{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \quad \text{exists.}$$

Note that if $K \subset \mathbf{N}$ is a finite set, then $\delta(K) = 0$, and for any set $K \subset \mathbf{N}$, $\delta(K^C) = 1 - \delta(K)$.

Definition 3.1 A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$,

$$\delta\bigl(\bigl\{k \in \mathbf{N} : |x_k - \ell| \geq \varepsilon\bigr\}\bigr) = 0.$$

In another direction, a new type of convergence called lacunary statistical convergence was introduced in [17] as follows. A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. A sequence (x_k) of real numbers is said to be lacunary statistically convergent to L (or, S_θ-convergent to L) if for any $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \bigl|\bigl\{k \in I_r : |x_k - L| \geq \epsilon\bigr\}\bigr| = 0,$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [17] the relation between lacunary statistical convergence and statistical convergence was established among other things.

We now have the following definitions.

Definition 3.2 (See [22, 25]) Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$. The sequence (x_k) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\epsilon > 0$, the set $A(\epsilon) = \{n \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in \mathcal{I}$. The class of all $\mathcal{I}$-statistically convergent sequences will be denoted by $S(\mathcal{I})$.

Definition 3.3 ([24]) Let θ be a lacunary sequence. A sequence $x = (x_k)$ is said to be $\mathcal{I}$-lacunary statistically convergent to L or $S_\theta(I)$-convergent to L if for any $\epsilon > 0$ and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \bigl|\bigl\{k \in I_r : |x_k - L| \geq \epsilon\bigr\}\bigr| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(S_\theta(\mathcal{I}))$. The class of all $\mathcal{I}$-lacunary statistically convergent sequences will be denoted by $S_\theta(\mathcal{I})$.

It can be checked, as in the case of statistically and lacunary statistically convergent sequences, that both $S(\mathcal{I})$ and $S_\theta(\mathcal{I})$ are linear subspaces of the space of all real sequences.

Remark 3.1 For $\mathcal{I} = \mathcal{I}_{fin} = \{A \subseteq \mathbf{N} : A \text{ is a finite subset}\}$, $\mathcal{I} - S_\theta$-convergence coincides with lacunary statistical convergence which is defined in [17].

Let $E \subseteq \mathbf{N} \times \mathbf{N}$ be a two-dimensional set of positive integers and let $E_{m,n}$ be the numbers of (i,j) in K such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density $\delta_2(E)$ of E is defined as follows:

$$\liminf_{m,n} \frac{E_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $(\frac{E_{m,n}}{mn})_{m,n=1,1}^{\infty,\infty}$ has a limit, we say that E has a natural density and is defined as follows:

$$\lim_{m,n} \frac{E_{m,n}}{mn} = \delta_2(E).$$

For example, let $E = \{(i^2, j^2) : (i,j) \in \mathbf{N} \times \mathbf{N}\}$. Then

$$\delta_2(E) = \lim_{m,n} \frac{E_{m,n}}{mn} \leq \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(*i.e.*, the set E has double natural density zero).

Recently, Mursaleen and Edely [34] presented the notion of statistical convergence for a double sequence $x = (x_{kl})$ as follows:

A real double sequence $x = (x_{kl})$ is said to be statistically convergent to L provided that for each $\epsilon > 0$,

$$\lim_{m,n} \frac{1}{mn} \left| \left\{ (k,l) : k \leq m \text{ and } l \leq n, |x_{k,l} - L| \geq \epsilon \right\} \right| = 0.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0, \qquad h_r = k_r - k_{k-1} \to \infty \quad \text{as } r \to \infty$$

and

$$l_0 = 0, \qquad \bar{h}_s = l_s - l_{s-1} \to \infty \quad \text{as } s \to \infty.$$

Let us denote $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$ and $\theta_{r,s}$ is determined by $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\}$.

We have the following.

Definition 3.4 Let (x_{kl}) be a sequence in a locally solid Riesz space (L,τ). We say that x is $\mathcal{I}_{\theta_{r,s}}$-statistically-$\tau$-convergent to x_0 if for every τ-neighborhood U of zero and for $\delta > 0$,

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : x_{kl} - x_0 \notin U\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I}_{\theta_{r,s}} - st_\tau - \lim x_{kl} = x_0$ (or $x_{kl} \xrightarrow{\mathcal{I}_{\theta_{r,s}} - st_\tau} x_0$ in brief).

Remark 3.2 For $\mathcal{I} = \mathcal{I}_{fin}$, $\mathcal{I}_{\theta_{r,s}}$-statistical-$\tau$-convergence becomes double lacunary statistical τ-convergence in a locally solid Riesz space.

Definition 3.5 Let (x_{kl}) be a sequence in a locally solid Riesz space (L,τ). We say that x is $\mathcal{I}_{\theta_{rs}}$-statistically-$\tau$-bounded if for every τ-neighborhood U of zero and $\delta > 0$, there exists $\alpha > 0$ such that

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : \alpha x_{kl} \notin U\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Definition 3.6 Let (x_{kl}) be a sequence in a locally solid Riesz space (L,τ). We say that x is $\mathcal{I}_{\theta_{r,s}}$-statistically-$\tau$-Cauchy if for every τ-neighborhood U of zero and $\delta > 0$, there exist $p,q \in \mathbb{N}$ such that

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : x_{kl} - x_{pq} \notin U\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Now we are ready to present some basic properties of this new convergence in a locally solid Riesz space.

Theorem 3.1 *Let (L,τ) be a Hausdorff locally solid Riesz space, $x = (x_{kl})$ and $y = (y_{kl})$ be two sequences in L. Then the following hold*:

(a) *If $\mathcal{I}_{\theta_{r,s}} - st_\tau - \lim x_{kl} = y_0$ and $\mathcal{I}_{\theta_{r,s}} - st_\tau - \lim x_{kl} = z_0$, then $y_0 = z_0$.*
(b) *If $\mathcal{I}_{\theta_{r,s}} - st_\tau - \lim x_{kl} = x_0$, then $\mathcal{I}_{\theta_{r,s}} - st_\tau - \lim \alpha x_{kl} = \alpha x_0$ for each $\alpha \in \mathbb{R}$.*
(c) *If $\mathcal{I}_{\theta_{r,s}} - st_\tau - \lim x_{kl} = x_0$ and $\mathcal{I}_{\theta_{r,s}} - st_\tau - \lim y_{kl} = y_0$, then $\mathcal{I}_{\theta_{r,s}} - st_\tau - \lim(x_{kl} + y_{kl}) = x_0 + y_0$.*

Proof (a) Let U be any τ-neighborhood of zero. Then there exists a $V \in \mathcal{N}_{\mathrm{sol}}$ such that $V \subset U$. Take a $W \in \mathcal{N}_{\mathrm{sol}}$ such that $W + W \subset V$. Let $\delta = \frac{1}{5}$. Since $\mathcal{I}_{\theta_{r,s}} - st_\tau - \lim x_{kl} = y_0$ and $I_{\theta_{r,s}} - st_\tau - \lim x_{kl} = z_0$, we write

$$K_1 = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : x_{kl} - y_0 \notin W\right\}\right| < \delta\right\} \in F(\mathcal{I})$$

and

$$K_2 = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : x_{kl} - z_0 \notin W\right\}\right| < \delta\right\} \in F(\mathcal{I}).$$

Then $K = K_1 \cap K_2 \in F(\mathcal{I})$ and for $r,s \in K$,

$$\frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : x_{kl} - y_0 \notin W\right\}\right| < \delta,$$

i.e.,

$$\frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : x_{kl} - y_0 \in W\right\}\right| > 1 - \delta = \frac{4}{5}. \tag{1}$$

Similarly,

$$\frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : x_{kl} - z_0 \in W\right\}\right| > \frac{4}{5}. \tag{2}$$

Now write that $\{(k,l) \in I_{rs} : x_{(k,l)} - y_0 \in W\}$ and $\{(k,l) \in I_{rs} : x_{kl} - z_0 \in W\}$ cannot be disjoint, for then we will have $\frac{1}{h_{rs}}|\{(k,l) \in I_{rs}\}| > \frac{8}{5}$, which is impossible. So, there is a $(k_r, l_s) \in I_{rs}$ for which

$$x_{k_r,l_s} - y_0 \in W \quad \text{and} \quad x_{k_r,l_s} - z_0 \in W.$$

Then

$$x_0 - z_0 = y_0 - x_{k_r l_s} + x_{k_r l_s} - z_0 \in W + W \subset V \subset U.$$

Thus $y_0 - z_0 \in U$ for every τ-neighborhood U of zero. Since (L,τ) is Hausdorff, the intersection of all τ-neighborhoods of zero is the singleton $\{\theta_{r,s}\}$, and so $y_0 - z_0 = \theta$, *i.e.*, $y_0 = z_0$.

(b) Let $\mathcal{I}_{\theta_{r,s}} - st_\tau - \lim x_k = x_0$ and let U be an arbitrary τ-neighborhood of zero. Choose $V \in \mathcal{N}_{\mathrm{sol}}$ such that $V \subset U$. For any $1 > \delta > 0$,

$$K = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : x_{kl} - x_0 \notin V\right\}\right| < \delta\right\} \in F(\mathcal{I}),$$

i.e., $\forall r,s \in K$,

$$\frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : x_{kl} - x_0 \in V\right\}\right| > 1 - \delta.$$

First let $|\alpha| \leq 1$. Since V is balanced, $x_{kl} - x_0 \in V$ implies that $\alpha(x_{kl} - x_0) \in V$. Therefore

$$\left\{(k,l) \in I_{rs} : \alpha x_{kl} - \alpha x_0 \in V\right\} \supset \left\{(k,l) \in I_{rs} : x_{kl} - x_0 \in V\right\},$$

and so $\forall r,s \in K$,

$$\frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : \alpha x_{kl} - \alpha x_0 \in V\right\}\right| \geq \frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : x_{kl} - x_0 \in V\right\}\right| > 1 - \delta,$$

which implies that

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : \alpha x_{kl} - \alpha x_0 \notin V\right\}\right| < \delta\right\} \supset K$$

and finally

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : \alpha x_{kl} - \alpha x_0 \notin V\right\}\right| < \delta\right\} \in F(\mathcal{I}).$$

If $|\alpha| > 1$ and $[|\alpha|]$ is the smallest integer greater or equal to $|\alpha|$, choose $W \in \mathcal{N}_{\mathrm{sol}}$ such that $[|\alpha|]W \subset V$. Again, for $1 > \delta > 0$, taking

$$K = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : x_{kl} - x_0 \notin W\right\}\right| < \delta\right\} \in F(\mathcal{I})$$

and in view of the fact that

$$|\alpha x_0 - \alpha x_{kl}| = |\alpha||x_0 - x_{kl}| \leq \left[|\alpha|\right]|x_{nm} - x_0| \in \left[|\alpha|\right]W \subset V \subset Um,$$

which implies that $\alpha x_0 - \alpha x_{kl} \in V \subset U$, proceeding as before, we conclude that

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : \alpha x_{kl} - \alpha x_0 \notin U\right\}\right| < \delta\right\} \in F(\mathcal{I}).$$

This proves that $I_{\theta_{r,s}} - st_\tau - \lim \alpha x_{kl} = \alpha x_0$.

(c) Let U be an arbitrary τ-neighborhood of zero. Then there are $V, W \in \mathcal{N}_{\mathrm{sol}}$ such that $W + W \subset V \subset U$. Since $I_\theta - st_\tau - \lim x_{kl} = x_0$ and $I_{\theta_{r,s}} - st_\tau - \lim y_{kl} = y_0$, we get, for $0 < \delta < 1$,

$$K_1 = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : x_{kl} - x_0 \notin W\right\}\right| < \frac{\delta}{3}\right\} \in F(\mathcal{I})$$

and

$$K_2 = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : y_{kl} - y_0 \notin W\right\}\right| < \frac{\delta}{3}\right\} \in F(\mathcal{I}).$$

If $K = K_1 \cap K_2$, then $\forall r,s \in K$,

$$\frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : x_{kl} - x_0 \notin W\right\}\right| < \frac{\delta}{3},$$

i.e.,

$$\frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : x_{kl} - x_0 \in W\right\}\right| > 1 - \frac{\delta}{3}$$

and also

$$\frac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : y_{kl} - y_0 \notin W\right\}\right| < \frac{\delta}{3}.$$

But

$$(x_{kl} + y_{kl}) - (x_0 + y_0) = (x_{kl} - x_0) + (y_{kl} - y_0) \in W + W \subset V \subset U$$

$\forall (k,l) \in I_{rs}$ such that $k,l \in A \cap B$ when $\{(k,l) \in I_{rs} : x_{kl} - x_0 \in W\} = A$ (say) and $\{(k,l) \in I_{rs} : y_{kl} - y_0 \in W\} = B$ (say). Note that

$$|A| = |A \cap B| + |A \backslash B| \leq |A \cap B| + \left|B^c\right|,$$

i.e.,

$$\frac{1}{h_{rs}}|A| \le \frac{1}{h_{rs}}|A\cap B| + \frac{1}{h_{rs}}\left|B^c\right| < \frac{1}{h_{rs}}|A\cap B| + \frac{\delta}{3},$$

i.e.,

$$\begin{aligned}\frac{1}{h_{rs}}|A\cap B| &= \frac{1}{h_{rs}}\left|\left\{(k,l)\in I_{rs} : x_{kl}-x_0 \in W \wedge y_{kl}-y_0 \in W\right\}\right| \\ &> \frac{1}{h_{rs}}\left|\left\{(k,l)\in I_{rs} : x_{kl}-x_0 \in W\right\}\right| - \frac{\delta}{3} \\ &> 1-\frac{\delta}{3}-\frac{\delta}{3} \\ &> 1-\delta.\end{aligned}$$

Since

$$\left\{(k,l)\in I_{rs} : (x_{kl}+y_{kl})-(x_0+y_0)\in U\right\} \supset A\cap B,$$

so for all $r,s\in K$,

$$\frac{1}{h_{rs}}\left|\left\{(k,l)\in I_{rs} : (x_{kl}+y_{kl})-(x_0+y_0)\in U\right\}\right| \ge \frac{1}{h_{rs}}|A\cap B| > 1-\delta,$$

i.e.,

$$\frac{1}{h_{rs}}\left|\left\{(k,l)\in I_{rs} : (x_{kl}+y_{kl})-(x_0+y_0)\notin U\right\}\right| < \delta.$$

Hence

$$K \subset \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}}\left|\left\{(k,l)\in I_{rs} : (x_{kl}+y_{kl})-(x_0+y_0)\notin U\right\}\right| < \delta\right\}$$

and so

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}}\left|\left\{(k,l)\in I_{rs} : (x_{kl}+y_{kl})-(x_0+y_0)\in U\right\}\right| < \delta\right\} \in F(\mathcal{I}).$$

This completes the proof of the theorem. □

Theorem 3.2 *Let (L,τ) be a locally solid Riesz space. Let $x=\{x_{kl}\}$, $y=\{y_{kl}\}$ and $z=\{z_{kl}\}$ be three sequences in L such that $x_{kl}\le y_{kl}\le z_{kl}$ for each $(k,l)\in\mathbb{N}\times\mathbb{N}$. If $\mathcal{I}_{\theta_{r,s}}-st_\tau-\lim x_{kl} = a = \mathcal{I}_{\theta_{r,s}}-st_\tau-\lim z_{nm}$, then $\mathcal{I}_{\theta_{r,s}}-st_\tau-\lim y_{kl}=a$.*

Proof Let U be an arbitrary τ-neighborhood of zero. Take $V,W\in\mathcal{N}_{\mathrm{sol}}$ such that $W+W\subset V\subset U$. Since $\mathcal{I}_{\theta_{r,s}}-st_\tau-\lim x_{kl}=a=\mathcal{I}_{\theta_{r,s}}-st_\tau-\lim z_{kl}$, so for $0<\delta<1$,

$$K_1 = \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}}\left|\left\{(k,l)\in I_{rs} : x_{kl}-a\notin W\right\}\right| < \frac{\delta}{3}\right\} \in F(\mathcal{I})$$

and

$$K_2 = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : z_{kl} - a \notin W\right\} \right| < \frac{\delta}{3}\right\} \in F(\mathcal{I}).$$

Hence we observe that $\forall r, s \in K$,

$$\frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : x_{kl} - a \notin W\right\} \right| < \frac{\delta}{3},$$

i.e.,

$$\frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : x_{kl} - a \in W\right\} \right| > 1 - \frac{\delta}{3}$$

and

$$\frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : z_{kl} - a \notin W\right\} \right| < \frac{\delta}{3}.$$

Writing $A = \{(k,l) \in I_{rs} : x_{kl} - a \in W\}$ and $B = \{(k,l) \in I_{rs} : z_{kl} - a \in W\}$, we see that $\forall k, l \in A \cap B$,

$$\begin{aligned}
&x_{kl} \leq y_{kl} \leq z_{kl}, \\
&x_{kl} - a \leq y_{kl} - a \leq z_{kl} - a, \\
&|y_{kl} - a| \leq |x_{kl} - a| + |z_{kl} - a| \in W + W \subset V,
\end{aligned}$$

and as V is solid, so

$$y_{kl} - a \in V \subset U.$$

Clearly, $\{(k,l) \in I_{rs} : y_{kl} - a \in U\} \supset A \cap B$ and as in the previous theorem, we show that $\forall r, s \in K$,

$$\frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : y_{kl} - a \in U\right\} \right| \geq \frac{1}{h_{rs}} |A \cap B| > 1 - \delta,$$

i.e.,

$$\frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : y_{kl} - a \notin U\right\} \right| < \delta.$$

Hence

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : y_{kl} - a \notin U\right\} \right| < \delta\right\} \supset K,$$

where $K \in F(\mathcal{I})$ and so

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left| \{k \in I_{rs} : y_{kl} - a \notin U\} \right| \geq \delta\right\} \in \mathcal{I}.$$

This proves that $\mathcal{I}_{\theta_{r,s}} - st_\tau - \lim y_{kl} = a$. This completes the proof of the theorem. □

Theorem 3.3 *An $\mathcal{I}_{\theta_{r,s}}$-statistically τ-convergent sequence (x_{kl}) in a locally solid Riesz space (L,τ) is $\mathcal{I}_{\theta_{r,s}}$-statistically τ-bounded.*

Proof Let (x_{kl}) be $I_{\theta_{r,s}}$-statistically τ-convergent to $x_0 \in L$. Let U be an arbitrary τ-neighborhood of zero. Choose $V, W \in \mathcal{N}_{\mathrm{sol}}$ such that $W + W \subset V \subset U$. Since W is absorbing, there is a $\mu > 0$ such that $\mu x_0 \in W$. Choose $\alpha \leq 1$ so that $\alpha \leq \mu$. Since W is solid and $|\lambda x_0| \leq |\mu x_0|$, we have $\alpha x_0 \in W$. Again, as W is balanced, $x_{kl} - x_0 \in W$ implies that $\alpha(x_{nm} - x_0) \in W$. Now, for any $0 < \delta < 1$,

$$K = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{rs} : x_{kl} - x_0 \notin W\right\}\right| < \delta\right\} \in F(\mathcal{I}).$$

Thus, for all $r,s \in K$,

$$\frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{rs} : x_{kl} - x_0 \notin W\right\}\right| < \delta,$$

i.e.,

$$\frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{rs} : x_{kl} - x_0 \in W\right\}\right| > 1 - \delta.$$

If $B_{rs} = \{(k,l) \in I_{rs} : x_{kl} - x_0 \in W\}$, then $\forall k,l \in B_{rs}$

$$\alpha x_{kl} = \alpha(x_{kl} - x_0) + \alpha x_0 \in W + W \subset V \subset U,$$

and so, for all $r,s \in K$,

$$\begin{aligned} \frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{rs} : \alpha x_{kl} \in W\right\}\right| &\geq \frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{rs} : x_{kl} - x_0 \in W\right\}\right| \\ &> 1 - \delta \end{aligned}$$

i.e.,

$$\frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{rs} : \alpha x_{kl} \notin W\right\}\right| < \delta.$$

Hence

$$K \subset \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{rs} : \alpha x_{kl} \notin W\right\}\right| < \delta\right\}.$$

Since $K \in F(\mathcal{I})$, so the set on the right-hand side also belongs to $F(\mathcal{I})$ and this proves that (x_{kl}) is $\mathcal{I}_{\theta_{r,s}}$-statistically τ-bounded. □

Theorem 3.4 *If a sequence (x_{kl}) in a locally solid Riesz space (L,τ) is $\mathcal{I}_{\theta_{r,s}}$-statistically τ-convergent, then it is $\mathcal{I}_{\theta_{r,s}}$-statistically τ-Cauchy.*

Proof Let (x_{kl}) be $\mathcal{I}_{\theta_{r,s}}$-statistically τ-convergent to $x_0 \in L$. Let U be an arbitrary τ-neighborhood of zero. Choose $V, W \in \mathcal{N}_{\mathrm{sol}}$ such that $W + W \subset V \subset U$. Let $0 < \delta < 1$.

Therefore

$$K = \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : x_{kl} - x_0 \notin W\right\} \right| < \delta \right\} \in F(\mathcal{I}).$$

For all $r, s \in K$,

$$\frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : x_{kl} - x_0 \notin W\right\} \right| < \delta,$$

i.e.,

$$\frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : x_{kl} - x_0 \in W\right\} \right| > 1 - \delta.$$

Take $r, s \in K$ and in view of the above, we can choose $p, q \in \{(k,l) \in I_{rs} : x_{kl} - x_0 \in W\}$ (since this set cannot be empty). Then $x_{pq} - x_0 \in W$. Now observe that if for $(k,l) \in I_{rs}, x_{kl} - x_0 \in W$, then

$$x_{kl} - x_{pq} = x_{kl} - x_0 + x_0 - x_{pq} \in W + W \subset V \subset U.$$

Hence, as in the earlier proofs, we can prove that

$$K \subset \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left| \left\{(k,l) \in I_{rs} : x_{kl} - x_{pq} \notin W\right\} \right| < \delta \right\},$$

which consequently implies that (x_{kl}) is $\mathcal{I}_{\theta_{r,s}}$-statistically τ-Cauchy.

This completes the proof of the theorem. □

It should be noted that single and double case of I_λ-statistical convergence in locally solid Riesz spaces are introduced in [35] and [36] respectively.

Competing interests

The author declares that they have no competing interests.

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